

# Duality of $Sp(2N_c)$ and $SO(N_c)$ Supersymmetric Gauge Theories with Adjoint Matter

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## ABSTRACT

We discuss electric-magnetic duality in two new classes of supersymmetric Yang-Mills theories. The models have gauge group  $Sp(2N_c)$  or  $SO(N_c)$  with matter in both the adjoint and defining representations. By perturbing these theories with various superpotentials, we find a variety of new infrared fixed points with dual descriptions. This work is complementary to that of Kutasov and Schwimmer on  $SU(N_c)$  and of Intriligator on other models involving  $Sp(2N_c)$  and  $SO(N_c)$ .

## 1. Introduction

Recent developments in the study of supersymmetric Yang-Mills theories have led to the discovery of a wide variety of phenomena and have given us insight into many aspects of strongly coupled physics [1-13]. Among these phenomena are large classes of superconformally invariant infrared fixed points [5]. In recent months, the study of these four-dimensional theories has led to a realization of their rich structure.

Perhaps the most exciting property of supersymmetric Yang-Mills theories is that they can be shown to have a form of electric-magnetic duality [4-12]. The study of duality in  $N=1$  models has just begun — only a few simple classes of models have been investigated. It is important to explore other possibilities; with the elucidation of many dual models, a pattern may become apparent, leading eventually to an understanding of the general principles involved.

In this paper, we consider Yang-Mills theories with gauge group  $Sp(2N_c)$  and  $SO(N_c)$  with matter in the adjoint representation as well as the defining representation. These models are very similar to those recently studied in Refs. [6,10] and are complementary to those studied in Ref. [12]. Because of these similarities, we follow closely the exposition given in [10]. In our conclusion we comment on possible relations between the duality in these models and that of finite  $N=2$  theories [5,7,10].

## 2. $Sp(2N_c)$ with an adjoint and fundamentals

### 2.1. Preliminaries

We consider an  $Sp(2N_c)$  supersymmetric Yang-Mills theory<sup>1</sup> with an adjoint field  $X$  (which may be taken as a symmetric 2-tensor) and  $2N_f$  fundamentals  $Q^i$ . The coefficient of the one-loop  $\beta$ -function is  $b_0 = 2(N_c + 1) - N_f$ . For some values of  $N_f$ , we expect that at the origin of moduli space there is an interacting conformal field theory in the infrared, which may possess dual descriptions. As yet, we do not understand the duality of this model for arbitrary superpotential. However, the description is straightforward if

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<sup>1</sup> Note that Refs. [12],[5],[11] refer to this group as  $Sp(N_c)$ .

we include a superpotential

$$W = \lambda_{k+1} \frac{1}{2(k+1)} \text{tr} X^{2(k+1)}. \quad (2.1)$$

Traces are taken with the invariant tensor  $J$  which may be thought of as the matrix  $\mathbb{1}_{N_c} \otimes i\sigma_2$ . The flavor group of these models is  $SU(2N_f) \times U(1)_R$ ; the matter fields transform as:

$$\begin{aligned} Q &\in \left( \mathbf{2N_f}, 1 - \frac{N_c + 1}{N_f(k+1)} \right) \\ X &\in \left( \mathbf{1}, \frac{1}{k+1} \right). \end{aligned} \quad (2.2)$$

We may form the following gauge invariant polynomials in these fields:

$$\begin{aligned} u_n &= \frac{1}{2n} \text{tr} X^{2n} \\ M_{(2n)}^{rs} &= Q^r X^{2n} Q^s \\ M_{(2n+1)}^{rs} &= Q^r X^{2n+1} Q^s \end{aligned} \quad (2.3)$$

where  $n = 0, 1, 2, \dots$ . The mesons  $M_{2n}$  transform under the  $N_f(2N_f - 1)$ -dimensional antisymmetric tensor representation of  $SU(2N_f)$  while  $M_{2n+1}$  resides in the  $N_f(2N_f + 1)$ -dimensional symmetric tensor representation. Of course, many of these are redundant operators. In the presence of the superpotential (2.1), the chiral ring consists of the operators  $u_n$  and  $M_{(2n)}$  for  $n = 0, 1, \dots, k$  and  $M_{(2n+1)}$  for  $n = 0, 1, \dots, k-1$ .

## 2.2. Stability

We wish to study under what circumstances this model possesses stable ground states. Consider deforming the model (2.1) to

$$W = \sum_{n=1}^{k+1} \lambda_n u_n. \quad (2.4)$$

The field  $X$  can be rotated into the  $2 \times 2$  block form  $X = \text{diag}(x_1 \sigma_1, x_2 \sigma_1, \dots)$ , where  $\sigma_1$  is a Pauli matrix. Given such a superpotential, one finds supersymmetric ground states for  $\langle Q \rangle = 0$  and  $x_i$  satisfying the equation

$$x \sum_{n=0}^k \lambda_{n+1} x^{2n} = x \prod_{j=1}^k (x^2 - \alpha_{(j)}^2) = 0. \quad (2.5)$$

Generically there are  $k + 1$  independent solutions of this equation (the overall sign of an eigenvalue is unobservable) which we will label  $\alpha_{(j)}$ ,  $j = 0, \dots, k$ , with  $\alpha_{(0)} = 0$ . A ground state will then be labeled by a set of integers  $(p_0, p_1, \dots, p_k)$ ; each  $p_j$  gives the number of eigenvalues  $x_i$  of  $\langle X \rangle$  which are equal to  $\alpha_{(j)}$ . The symmetry surviving in such a vacuum state is

$$Sp(2p_0) \times U(p_1) \times \dots \times U(p_k) \quad (2.6)$$

with  $\sum_{j=0}^k p_j = N_c$ . In these vacua, the adjoint fields will be massive and can be integrated out. The matter fields  $Q$  decompose as follows: there are  $2N_f$  flavors of fields in the fundamental and antifundamental representations of each of the unitary group factors, while there remain  $N_f$  flavors ( $2N_f$  fields  $Q^r$ ) in the symplectic factor. The resulting low-energy theory is thus a product of SQCD-like models; these are known to possess stable vacua for  $2N_f \geq p_j$ ,  $j > 0$  [15] and  $N_f \geq p_0 + 1$  [11]. We conclude that the theory with superpotential (2.4) has stable vacua for  $N_f \geq (N_c + 1)/(2k + 1)$ ; the same is true for the theory with superpotential (2.1), assuming the result varies smoothly for small coupling  $\lambda_n$ ,  $n = 1, \dots, k$ .

If the coefficients  $\lambda_n$  are properly tuned, some of the solutions  $\alpha_{(j)}$  will coincide. In this case the  $x_i$  satisfy the equation

$$x^{1+2r_0} \prod_{j=1}^m (x^2 - \alpha_{(j)}^2)^{r_j} = 0. \quad (2.7)$$

such that  $\sum r_j = k$ . The  $Sp(2p_0)$  group factor now has a massless adjoint superfield with an effective superpotential  $\text{Tr} X^{2(r_0+1)}$ ; similarly the  $U(p_j)$  factor corresponding to the  $p_j$  eigenvalues  $x_i = \alpha_{(j)}$  has an adjoint superfield with effective superpotential  $\text{Tr} X^{r_j+1}$ . Such  $U(N_c)$  theories were studied in [10].

### 2.3. The dual model

We will refer to the above  $Sp(2N_c)$  model as the electric theory. The dual magnetic theory is an  $Sp(2\tilde{N}_c)$  gauge theory with  $\tilde{N}_c = (2k + 1)N_f - N_c - 2$ . Under the duality transformation, the mesons  $M_{(2n)}$  and  $M_{(2n+1)}$  of (2.3) are mapped to gauge singlet fields which we will refer to by the same name. In addition the magnetic theory possesses

$2N_f$  fields  $q$  in the fundamental representation of  $Sp(2N_c)$  and an adjoint field  $Y$ . The superpotential of the magnetic theory is:

$$W_m = \lambda_{k+1} \frac{1}{2(k+1)} \text{tr} Y^{2(k+1)} + \sum_{n=0}^{2k} M_{(n)}^{rs} q_r Y^{2k-n} q_s. \quad (2.8)$$

The 't Hooft anomaly matching conditions for the global symmetries at the origin of moduli space are powerful constraints. The fields of the magnetic theory transform as follows under the  $SU(2N_f) \times U(1)_R$  flavor symmetry:

$$\begin{aligned} q &\in \left( \overline{\mathbf{2N_f}}, 1 - \frac{\tilde{N}_c + 1}{N_f(k+1)} \right), \\ Y &\in \left( \mathbf{1}, \frac{1}{k+1} \right), \\ M_{(2n)} &\in \left( \mathbf{N_f}(\mathbf{2N_f} - \mathbf{1}), 2 - \frac{2(N_c + 1) - 2nN_f}{N_f(k+1)} \right), \\ M_{(2n+1)} &\in \left( \mathbf{N_f}(\mathbf{2N_f} + \mathbf{1}), 2 - \frac{2(N_c + 1) - (2n+1)N_f}{N_f(k+1)} \right). \end{aligned}$$

The anomalies are found to be, in both the electric and magnetic theories:

$$\begin{aligned} U(1)_R : & - \frac{N_c(2N_c + 3)}{k+1} \\ U(1)_R^3 : & - 4 \frac{N_c(N_c + 1)^3}{N_f^2(k+1)^3} + N_c(2N_c + 1) \left( 1 - \left( \frac{k}{k+1} \right)^3 \right) \\ U(1)_R SU(2N_f)^2 : & - \frac{N_c(N_c + 1)}{N_f(k+1)} \\ SU(2N_f)^3 : & + 2N_c. \end{aligned} \quad (2.9)$$

#### 2.4. Deformations

We first consider the addition of a mass term for a quark in the electric theory. This perturbation will cause flow to a similar theory with one fewer flavor of quarks. In the dual theory, we expect, following the duality discussed above, to flow to a theory with gauge group  $Sp(2[\tilde{N}_c - 2k - 1])$ . Indeed, the mass perturbation  $mQ^{2N_f-1}Q^{2N_f}$  gives rise to a superpotential for the magnetic theory of the form:

$$W_m = \lambda_{k+1} \frac{1}{2(k+1)} \text{tr} Y^{2(k+1)} + \sum_{n=0}^{2k} M_{(n)}^{rs} q_r Y^{2k-n} q_s + m M_{(0)}^{2N_f-1, 2N_f}$$

The  $M$  equations of motion then imply the vacuum expectation values satisfy:

$$\langle q_{2N_f-1} Y^j q_{2N_f} \rangle = -m \delta_{j,2k} ; j = 0, 1, \dots, 2k \quad (2.10)$$

The  $D$ -flatness and other  $F$ -flatness conditions result in expectation values which break  $Sp(2\tilde{N}_c) \rightarrow Sp(2[\tilde{N}_c - 2k - 1])$ , as required. The fields  $q_{2N_f-1}, q_{2N_f}$  are eaten by the broken gauge multiplets. The adjoint field  $Y$  decomposes into an adjoint of the low energy gauge group, along with  $2k + 1$  flavors  $Z_a, Z'_a$ , of which  $2k$  are eaten by gauge multiplets; the remaining one becomes massive through the  $\text{Tr} Y^{2(k+1)} \rightarrow Z_1 \langle Y^{2k} \rangle Z'_1$  interaction. In the end, the number of flavors remaining is  $N_f - 1$ . (Some massive singlets are also generated.)

We can also study the effects of adding  $Q^{2N_f-1} X^j Q^{2N_f}$  for  $j = 1, \dots, 2k$  to the superpotential of the electric theory. In this case, the gauge group of the magnetic theory is reduced by  $2k - j + 1$  colors; the fields  $q_{2N_f-1}, q_{2N_f}$  are eaten by the broken gauge multiplets; the field  $Y$  decomposes into an adjoint of the low energy gauge group and  $2k - j + 1$  new flavors  $Z_a, Z'_a$ , of which  $2k - j$  are eaten by gauge multiplets and the remaining one develops an interaction  $Z_1 Y^j \langle Y^{2k-j} \rangle Z'_1$  from the  $\text{Tr} Y^{2(k+1)}$  interaction. In the end, the number of flavors remaining is  $N_f - 1$ , with an extra flavor coupled to the adjoint through a  $\text{Tr} Z_1 Y^j Z'_1$  term. This corresponds to the type of duality studied in Ref. [9] in the case of an  $SU(N_c)$  gauge group.<sup>2</sup>

Lastly, we may consider perturbations by operators of the form  $u_j$ . We have given the general analysis of this situation in Section 2.2. A particularly simple example is  $j = 1$ , corresponding to a mass term for the field  $X$ . Let us first discuss the electric theory. For simplicity, we consider the case  $N_f \geq N_c - 1$ . In the presence of the mass term for  $X$ , the eigenvalues of  $X$  must satisfy

$$x(m + \lambda_k x^{2k}) = 0$$

The solutions,  $x = 0, \eta_i (-m/\lambda_{k+1})^{1/2k}$ , with  $\eta_i^{2k} = 1$ , constitute  $k + 1$  independent values. At scales well below the adjoint mass, we have an

$$Sp(2p_0) \times U(p_1) \dots \times U(p_k)$$

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<sup>2</sup> Thus the duality of Ref. [9], along with many generalizations, may be straightforwardly derived from that of Refs. [6],[10].

gauge symmetry, as was explained in Section 2.2, with  $\sum_{r=0}^k p_r = N_c$ . All the vacua are stable. In the dual theory, we also have a massive adjoint field, and the analysis proceeds similarly. However, in this case, in identifying the ground states, we must take care to check stability. It is convenient to write the symmetry group of the magnetic theory in the form:

$$Sp(2[(2k+1)N_f - N_c - 2]) \rightarrow Sp(2[N_f - j_0 - 2]) \times U(2N_f - j_1) \times \dots \times U(2N_f - j_k)$$

where  $\sum_{r=0}^k j_r = N_c$ . Each of these groups contains the appropriate magnetic  $Mqq$  coupling, where the singlet  $M$  is a linear combination of the  $M_{(j)}$  of the unbroken magnetic theory. If  $j_0 = -1$ , then the  $Sp(2[N_f + 1])$  theory confines and has no vacuum [12].<sup>3</sup> If  $j_r = 0$ ,  $r > 0$ , then the  $SU(2N_f)$  theory confines but possesses a vacuum in which its baryons  $B, \tilde{B}$  condense and break the remaining  $U(1)$ [10].<sup>4</sup> Thus, stability requires  $j_r \geq 0$  for all  $r$  and so there are the same number of stable ground states as in the electric theory. With  $p_r$  and  $j_r$  identified, it becomes clear that this duality is consistent with that discovered in [5] and explored further in [11].

It is interesting to consider a few specific examples. Suppose  $p_0 = 0$ . Then the electric gauge group is a product of  $U(p_i)$  factors. The magnetic group is  $Sp(2[N_f - 2])$  times a product of  $U(2N_f - p_i)$  factors. The  $Sp(2[N_f - 2])$  confines [11] and its mesons  $N = qq$  acquire mass through the coupling  $Mqq$  in the magnetic superpotential. The low energy magnetic  $Sp(2[N_f - 2])$  theory is therefore empty.

Conversely, if  $p_0 = N_f - 2$ , then the electric  $Sp(2[N_f - 2])$  factor confines into mesons  $M = QQ$  and develops a superpotential  $\text{Pf}M$  [11]. The dual magnetic group is a product of  $U(2N_f - p_i)$  factors, but a number of singlet meson fields are left over. An instanton effect in the broken magnetic  $Sp$  gauge group generates a term proportional to  $\text{Pf}M$  [5], [11] and preserves the duality.

Another example is when  $p_0 = N_c$ ; the field  $X$  becomes massive but the  $Sp(2N_c)$  gauge group is unbroken. In this case the magnetic theory has a factor  $Sp(2[N_f - N_c - 2])$

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<sup>3</sup> The field  $N_{rs} = (q_r q_s)$  has superpotential  $W = M^{rs} N_{rs}$  and a constraint  $\text{Pf}N = \Lambda_L^{2(N_c+1)}$ ; these are inconsistent.

<sup>4</sup> The field  $N_r^s = (q_r \tilde{q}^s)$  has superpotential  $W = M_s^r N_r^s$  and a constraint  $\det N - B\tilde{B} = \Lambda_L^{2N_c}$ ; these are consistent.

and several factors of  $U(2N_f)$ , each of which has  $2N_f$  flavors. As discussed in [10], the  $SU(2N_f)$  theories confine and their baryons condense, breaking the  $U(1)$  groups. The magnetic theory reduces to the  $Sp(2[N_f - N_c - 2])$  theory expected from [5,11].

One may also consider more general deformations as in Eq. (2.4). For generic values of the couplings, the analysis is similar to the above. At special values of the couplings, roots of  $W'(x) = 0$  may coincide, leading to low energy theories which have adjoint matter and superpotentials such as  $\text{Tr}X^r$ . As an example, consider the superpotential with  $W'(x) = x^{(2r+1)}(x^2 - a^2)^{k-r}$ . The low energy electric theory is

$$Sp(2p_0) \times U(N_c - p_0)$$

with superpotential  $\text{Tr}X_{Sp}^{2(r+1)} + \text{Tr}X_U^{k-r+1}$ . The magnetic theory has a low energy theory

$$Sp(2\tilde{p}_0) \times U(\tilde{N}_c - \tilde{p}_0)$$

and the same superpotential as the electric theory. But stability requires that

$$N_f \geq \frac{(\tilde{N}_c - \tilde{p}_0)}{2(k - r + 1)} ; N_f \geq \frac{(\tilde{p}_0 + 1)}{(2r + 1)} ;$$

the former relation follows from [10] and the latter from Section 2.2. These force

$$(2r + 1)N_f - N_c - 2 \leq \tilde{p}_0 \leq (2r + 1)N_f - 1 ,$$

showing that both theories have  $N_c + 1$  vacua. It can easily be checked that the duality illustrated here and in [10] is maintained by these vacua.

Thus, in a fashion very similar to that illustrated in Ref. [10], the deformations  $u_j$  lead in general to similar theories with lower values of  $k$ . The duality map is preserved under these deformations.

### 3. $SO(N_c)$ with an adjoint and vectors

#### 3.1. Preliminaries

We now consider  $SO(N_c)$  super Yang Mills theory with an adjoint field  $X$  (which may be thought of as the anti-symmetric 2-tensor) and  $N_f$  vectors  $Q^i$ . The one-loop  $\beta$ -function is  $b_0 = 2(N_c - 2) - N_f$ . We will again include a superpotential

$$W_e = \lambda_{k+1} \frac{1}{2(k+1)} \text{tr}X^{2(k+1)}. \quad (3.1)$$



Traces are taken with the Kronecker  $\delta$ . The flavor group of these models is  $SU(N_f) \times U(1)_R$ ; the matter fields transform as:

$$\begin{aligned} Q &\in \left( \mathbf{N}_f, 1 - \frac{N_c - 2}{N_f(k+1)} \right) \\ X &\in \left( \mathbf{1}, \frac{1}{k+1} \right). \end{aligned} \quad (3.2)$$

We may form the following gauge invariant polynomials in these fields:

$$\begin{aligned} u_n &= \frac{1}{2n} \text{tr} X^{2n} \\ M_{(2n)}^{rs} &= Q^r X^{2n} Q^s \\ M_{(2n+1)}^{rs} &= Q^r X^{2n+1} Q^s \end{aligned} \quad (3.3)$$

where  $n = 0, 1, 2, \dots$ . The mesons  $M_{2n}$  transform under the  $N_f(N_f + 1)/2$ -dimensional symmetric tensor representation of  $SU(N_f)$  while  $M_{2n+1}$  resides in the  $N_f(N_f - 1)/2$ -dimensional anti-symmetric tensor representation. In addition there are the generalized baryon operators obtained by contracting fields with an  $\epsilon$  tensor; we will not discuss them here.

Of course, many of these operators are redundant. In the presence of the superpotential (3.1), the chiral ring consists of the operators  $u_n$  and  $M_{(2n)}$  for  $n = 0, 1, \dots, k$  and  $M_{(2n+1)}$  for  $n = 0, 1, \dots, k-1$ , along with a number of generalized baryon operators.

### 3.2. Stability

We wish to study under what circumstances this model possesses stable ground states. Much of this analysis is carried over with little modification from the previous sections. Consider deforming the model (3.1) to

$$W = \sum_{j=1}^{k+1} \lambda_j u_j. \quad (3.4)$$

The field  $X$  can be rotated into the  $2 \times 2$  block form  $X = \text{diag}(x_1 i\sigma_2, \dots, x_{n_c} i\sigma_2)$  for  $SO(2n_c)$  and  $X = \text{diag}(x_1 i\sigma_2, \dots, x_{n_c} i\sigma_2, 0)$  for  $SO(2n_c + 1)$ , where  $\sigma_2$  is a Pauli matrix. Given such a superpotential, one finds supersymmetric ground states for  $\langle Q \rangle = 0$  and  $x_i$  satisfying the equation

$$x \sum_{n=0}^k \lambda_n x^{2n} = x \prod_{j=1}^k (x^2 - \alpha_{(j)}^2) = 0.$$

Generically there are  $k + 1$  independent solutions of this equation which we will label  $\alpha_{(j)}$ ,  $j = 0, \dots, k$ , with  $\alpha_{(0)} = 0$ . A ground state will then be labeled by a set of integers  $(p_0, p_1, \dots, p_k)$ ; each  $p_j$  gives the number of eigenvalues  $x_i$  of  $\langle X \rangle$  which are equal to  $\alpha_{(j)}$ . The symmetry surviving in such a vacuum state is

$$\begin{aligned} SO(2n_c + 1) &\rightarrow SO(2p_0 + 1) \times U(p_1) \times \dots \times U(p_k) \\ \text{or} \\ SO(2n_c) &\rightarrow SO(2p_0) \times U(p_1) \times \dots \times U(p_k) \end{aligned} \tag{3.5}$$

with  $\sum_{j=0}^k p_j = n_c$ . In these vacua, the adjoint fields will be massive and can be integrated out; the matter fields  $Q$  decompose into  $N_f$  flavors in the fundamental and anti-fundamental representations of each of the unitary group factors, plus  $N_f$  vectors in the orthogonal group factor. The resulting low-energy theory is thus a product of SQCD-like models; the unitary factors have stable vacua for  $N_f \geq p_j$ ,  $j > 0$  [15] while  $SO(N_c)$  has stable vacua for  $N_f \geq N_c - 4$  [5], [8]. We conclude that the theory with superpotential (3.4) has stable vacua for  $N_f \geq (N_c - 4)/(2k + 1)$ ; the same is true for the theory with superpotential (3.1), assuming the result varies smoothly for small couplings  $\lambda_n$ ,  $n = 1, \dots, k$ .

Again, when eigenvalues  $\alpha_{(j)}$  coincide the low energy theory has massless adjoint matter with a superpotential. We omit the details, since they are completely analogous to the previous case.

### 3.3. The dual model

The above  $SO(N_c)$  model will be referred to as the electric theory. The dual magnetic theory is an  $SO(\tilde{N}_c)$  gauge theory with  $\tilde{N}_c = (2k + 1)N_f - N_c + 4$ . Under the duality transformation, the mesons  $M_{(2n)}$  and  $M_{(2n+1)}$  of (2.3) are mapped to gauge singlet fields which we will refer to by the same name. In addition the magnetic theory possesses  $N_f$  fields  $q$  in the vector representation of  $SO(N_c)$  and an adjoint field  $Y$ . The superpotential of the magnetic theory is:

$$W_m = \lambda_{k+1} \frac{1}{2(k+1)} \text{tr} Y^{2(k+1)} + \sum_{n=0}^{2k} M_{(n)}^{rs} q_r Y^{2k-n} q_s. \tag{3.6}$$

Let us check the 't Hooft anomaly matching conditions for the global symmetries at the origin of moduli space. The fields of the magnetic theory transform as follows under the  $SU(N_f) \times U(1)_R$  flavor symmetry:

$$\begin{aligned}
q &\in \left( \overline{\mathbf{N}}_{\mathbf{f}}, 1 - \frac{\tilde{N}_c - 2}{N_f(k+1)} \right), \\
Y &\in \left( \mathbf{1}, \frac{1}{k+1} \right), \\
M_{(2n)} &\in \left( \mathbf{N}_{\mathbf{f}}(\mathbf{N}_{\mathbf{f}} + \mathbf{1})/2, 2 - \frac{2(N_c - 2) - 2nN_f}{N_f(k+1)} \right), \\
M_{(2n+1)} &\in \left( \mathbf{N}_{\mathbf{f}}(\mathbf{N}_{\mathbf{f}} - \mathbf{1})/2, 2 - \frac{2(N_c - 2) - (2n+1)N_f}{N_f(k+1)} \right).
\end{aligned}$$

The anomalies are found to be, in both the electric and magnetic theories:

$$\begin{aligned}
U(1)_R : & -\frac{N_c(N_c - 3)}{2(k+1)} \\
U(1)_R^3 : & -\frac{N_c(N_c - 2)^3}{N_f^2(k+1)^3} + \frac{N_c(N_c - 1)}{2} \left( 1 - \left( \frac{k}{k+1} \right)^3 \right) \\
U(1)_R SU(N_f)^2 : & -\frac{N_c(N_c - 2)}{N_f(k+1)} \\
SU(N_f)^3 : & +N_c.
\end{aligned} \tag{3.7}$$

### 3.4. Deformations

We first consider the addition of a mass term for a quark in the electric theory. This perturbation will cause flow to a similar theory with one fewer flavor of quarks. The mass perturbation  $mQ^{N_f}Q^{N_f}$  gives rise to a superpotential for the magnetic theory of the form

$$W_m = \lambda_{k+1} \frac{1}{2(k+1)} \text{tr} Y^{2(k+1)} + \sum_{n=0}^{2k} M_{(n)}^{rs} q_r Y^{2k-n} q_s + m M_{(0)}^{N_f, N_f}$$

The  $M$  equations of motion then imply the vacuum expectation values satisfy:

$$\langle q_{N_f} Y^j q_{N_f} \rangle = -m \delta_{j, 2k}; \quad j = 0, 1, \dots, 2k \tag{3.8}$$

The resulting expectation values for the individual fields break  $SO(\tilde{N}_c) \rightarrow SO(\tilde{N}_c - 2k - 1)$  as required, and there are  $N_f - 1$  massless vectors remaining.

The addition of operators  $QX^jQ$  for  $j = 1, \dots, 2k$  to the superpotential of the electric theory, in precise analogy to the symplectic case discussed above, causes the gauge group of the magnetic theory to be reduced and a certain superpotential to be induced in the low energy magnetic theory.

Now consider perturbations by operators of the form  $u_j$ . We have given the general analysis of this situation in Section 3.2. For  $j = 1$  the eigenvalues of  $X$  must satisfy

$$x(m + \lambda_k x^{2k}) = 0$$

as in the  $Sp(2N_c)$  case. At scales well below the adjoint mass, we have an

$$SO\left(N_c - 2\sum_{r=1}^k p_r\right) \times U(p_1) \dots \times U(p_k)$$

gauge symmetry, with  $2\sum_{r=1}^k p_r \leq N_c$ , as was explained in Section 3.2. In the dual theory, we also have a massive adjoint field, and the analysis proceeds similarly. However, we need to again consider stability. It is convenient to write the symmetry group of the magnetic theory in the form:

$$SO([2k+1]N_f - N_c + 4) \rightarrow SO\left(N_f - N_c + 4 + 2\sum_{r=1}^k j_r\right) \times U(N_f - j_1) \times \dots \times U(N_f - j_k) .$$

Stability requires  $j_r \geq 0$  and  $N_f + 4 \geq N_f - N_c + 4 + 2\sum_{r=1}^k j_r$ , and so the stable ground states are in one-to-one correspondence with those of the electric theory. Identifying  $p_r = j_r$  makes it clear that the duality of [5,8] is maintained.

One may also consider deformations by  $\text{tr}X^r$  for  $r > 2$ . The analysis is similar to the  $Sp(2N_c)$  case studied earlier and we omit the details. The duality map reduces in all cases in the appropriate fashion.

#### 4. Comments and Conclusions

Hints of relations between N=2 duality and that of N=1 have been noted by several authors [5,7,8,10]. In [7] we conjectured a specific relation between the two (which has since been verified [16]) that explains the appearance of the singlet fields  $M$  under N=1

duality. This relationship appears to be present in the models of [6] and [10] and persists in the models of this paper; however the models of [12] appear to show that the duality of N=1 models is not restricted to those which can be derived from N=2 theories.

Still, a general pattern has emerged. All of the known classes of models contain certain special theories whose duals have the same gauge group. The key feature of these self-dual theories is that they have marginal operators and associated lines of conformal fixed points [7] which connect the electric theory to a theory which is isomorphic, up to a reflection in the flavor group, to the magnetic theory. This phenomenon is exactly that of certain N=2 finite models, where the marginal coupling constant is the gauge coupling; the electric theory at  $g$  is dual to the magnetic theory at  $1/g$ , which itself is isomorphic (up to a flavor group reflection) to the electric theory at  $1/g$ .

The models of Kutasov and Schwimmer [6,10] have several properties which relate them to N=2.<sup>5</sup> The self-dual points in the  $SU(N_c)$  theories are found by beginning with the finite N=2 theory with  $2N_c$  hypermultiplets, whose gauge coupling is marginal. Using N=2 violating mass terms to remove all but  $1/k$  of the quarks, a low energy theory is generated with a marginal superpotential [7] of the form  $W = hQ_r X^k \tilde{Q}^r$ . The relevant perturbation by  $\text{Tr} X^{k+1}$  apparently drives the theory to a new fixed point with a marginal superpotential  $W = \frac{\lambda}{k+1} \text{Tr} X^{k+1} + hQ^r X^k \tilde{Q}_r$ . In the small  $h$  limit the model becomes the electric theory of [10]. The equation of motion for  $X$ , multiplied by  $Q^s$  and  $\tilde{Q}_s$ , gives

$$hQ^r X^k \tilde{Q}_r = -\frac{h^2}{\lambda} \sum_{n=0}^{k-1} (Q^r X^j \tilde{Q}_s)(Q^r X^{k-j-1} \tilde{Q}_s) + \dots$$

This suggests that one should introduce auxiliary mesons as in [7] and replace this operator with

$$\sum_{n=0}^{k-1} \left[ (N_j)_r^s (Q^r X^j \tilde{Q}_s) + \frac{\lambda}{4h^2} (N_j)_r^s (N_{k-j-1})_s^r \right] + \dots$$

Thus the large  $h$  limit, for which the  $N$ 's are massless, yields a model which is (roughly) isomorphic to the magnetic theory of [10], up to the usual reflection of the flavor group. For  $k = 1$  this argument is consistent with [7]. However, we do not as yet see how to turn this sketch into a rigorous derivation.

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<sup>5</sup> Many of the observations made below were also made in [10].

Nonetheless, these observations served as motivation to find the theories of the present paper, which again can be obtained by integrating out  $1/(2k+1)$  of the hypermultiplets of an  $N=2$  model, perturbing the theory by  $\text{Tr} X^{2(k+1)}$ , and using the equation of motion as a guide to find the magnetic theory at the self-dual point. The mapping of operators is straightforward, except for baryons in  $SO(N_c)$  for which the  $N=2$  duality transformation is still unknown. The models of [12] are not derivable from  $N=2$  theories, but they have the same structure: the self-dual models have marginal operators that connect the electric theory to a model isomorphic to the magnetic theory.

Other comments made at the end of [10] also apply to our models. Unitarity of the theory requires certain operators to decouple; it is unclear how this occurs. The dimensions of operators in the theory without a superpotential are unknown; from the theories studied here, one may infer certain aspects of the behavior of  $\text{Tr} X^r$  as a function of  $N_f$  and  $N_c$ . It would also be interesting to study the presumably intricate behavior of the  $SO(N_c)$  models, many of which have dyonic as well as magnetic duals, which we have not addressed in this paper. Finally, let us note that the theory with  $N_f = 0$  is a theory of great interest since one can reach it from a pure  $N=2$  Yang-Mills theory.

It is evident that the models studied in Refs. [5,6,8,10,11,12], along with those of this paper, all lie in the same class. Certain patterns are beginning to become apparent, and we hope to study them further in the near future.

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